E. Matrix Representation of Spin Angular Momentum

- Simple for spin ($\because$ $s = 1/2$, $m_s = 1/2, -1/2$ only two values)
- Arbitrarily, choose a matrix representation that the matrix for S_z is diagonal

$\left[\begin{array}{l} S_z \text{ has two eigenvalues: } \frac{\hbar}{2}, -\frac{\hbar}{2} \\ S_z \text{ is diagonal} \end{array}\right.$

(16) $$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\nearrow$ eigenvalue $\frac{\hbar}{2} \leftrightarrow$ eigenvector (eigenstate) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\searrow$ eigenvalue $-\frac{\hbar}{2} \leftrightarrow$ eigenvector (eigenstate) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$ $|1/2, 1/2\rangle = |m_s = 1/2\rangle = |1/2\rangle = |\uparrow\rangle = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in this representation

$|1/2, -1/2\rangle = |m_s = -1/2\rangle = |-1/2\rangle = |\downarrow\rangle = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in this representation

How about $[S_x]$, $[S_y]$ matrices?

- Conditions: $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$, $[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x$, $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$

$$\underbrace{\hat{S}^2}_{2\times 2} = \underbrace{\hat{S}_x^2}_{2\times 2} + \underbrace{\hat{S}_y^2}_{2\times 2} + \underbrace{\hat{S}_z^2}_{2\times 2}$$ has only $\frac{3}{4}\hbar^2$ as eigenvalue

(17) $$\hat{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ two eigenvalues ($\frac{3}{4}\hbar^2$ and $\frac{3}{4}\hbar^2$)

(18) $$\boxed{\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{work!}}$$

(16) $$\boxed{\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Eqs. (16), (18) give the most economical way of representing $s = 1/2$ angular momentum.

Ex: Check all commutation relations

Ex: Eigenvalues of $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

[<u>Meaning</u>: Measure any component (along any direction, not only x, y, z directions), only two possible outcomes of $+\frac{\hbar}{2}$, $-\frac{\hbar}{2}$]

- Eigenstate of $\hat{S}_z$ ($\hat{S}_x$) is <u>NOT</u> an eigenstate of $\left\{\begin{matrix}\hat{S}_x \\ \hat{S}_y\end{matrix}\right\}$ $\left(\left\{\begin{matrix}\hat{S}_y \\ \hat{S}_z\end{matrix}\right\}\right)$

Components <u>don't commute</u> (<u>don't</u> share eigenstates)

Pauli Matrices

$$\hat{S}_x = \frac{\hbar}{2}[\sigma_x] \; ; \; \hat{S}_y = \frac{\hbar}{2}[\sigma_y] \; ; \; \hat{S}_z = \frac{\hbar}{2}[\sigma_z] \tag{19}$$

$$\underbrace{[\sigma_x] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [\sigma_y] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad [\sigma_z] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{Pauli}^{\dagger}\text{ spin matrices (or Pauli matrices)}} \tag{20}$$

- $[\sigma]$'s properties are keys to spin physics
- $[\sigma]$'s are important in relativistic QM

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† Pauli was awarded the 1945 Nobel Physics Prize "for the discovery of the Exclusion Principle, also called the Pauli Principle". He did much more than that!

F. Spin is the best playground to learn Quantum Mechanics

- QM Spin AM is the simplest system that illustrates both the physics and mathematical structure of QM
- $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$, $\hat{S}^2$ in matrix forms satisfy the AM commutators

(a) Eigenvalues and Eigenstates

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_z \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \overbrace{\frac{\hbar}{2}}^{\text{eigenvalue}} \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{\text{eigenstate}}$$

$$\hat{S}_z \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underbrace{-\frac{\hbar}{2}}_{\text{eigenvalue}} \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{eigenstate}} \qquad (21)$$

This can be obtained by inspection.

Notations

	eigenvalue	eigenstate (eigenvector)
$\hat{S}_z$	$+\frac{\hbar}{2}$	$\underbrace{\left\|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle_z = \chi_{\frac{1}{2}\frac{1}{2}} = \left\|m_s = \tfrac{1}{2}\right\rangle = \left\|\uparrow\right\rangle_z = \overbrace{\alpha_z \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}}}_{\text{these are all the notations commonly found}}$
		$\uparrow s$ $\quad \uparrow m_s$ $\quad$ for $\hat{S}_{(z)}$
$\hat{S}_z$	$-\frac{\hbar}{2}$	$\left\|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle_z = \chi_{\frac{1}{2}-\frac{1}{2}} = \left\|m_s = -\tfrac{1}{2}\right\rangle = \left\|\downarrow\right\rangle_z = \underbrace{\beta_z \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$
		$\uparrow m_s$

They are orthogonal to each other

$$(1^* \quad 0^*) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \qquad \left[\text{c.f. } \int \phi_i^* \phi_j \, d\tau = 0 \ (i \neq j)\right]$$

Note: In our choice, $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is <u>diagonal</u>

[while $\hat{S}_x$, $\hat{S}_y$ are not]

- Doing so, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ play $\underbrace{\text{a special role}}_{\text{being simultaneous eigenstates of } \hat{S}_z \text{ \& } \hat{S}^2}$,
- $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are used as $\underbrace{\text{basis set of vectors}^{+}}_{\text{to express other vectors, e.g. } \begin{pmatrix} c \\ d \end{pmatrix}}$

$^{+}$ $\begin{pmatrix} c \\ d \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Analogy: choose $\hat{i}$ and $\hat{j}$ on a plane (there are many other choices), then a vector $\vec{V}$ is expressed as $\vec{V} = V_x\,\hat{i} + V_y\,\hat{j}$

How about $\hat{S}_x$?

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues of $[\sigma_x]$?

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$\therefore$ Eigenvalues of $\hat{S}_x$ are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

For each eigenvalue, then look for the eigenvector

E.g. $$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x}\begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{+1}_{\text{eigenvalue "+1" of } \sigma_x}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \left.\begin{matrix} a=b \\ b=a \end{matrix}\right\rangle \text{ same information}$$

$\therefore$ Eigenvector is $\begin{pmatrix} a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a fixed by normalization

<u>Normalization</u>: $|a|^2 (1^* \; 1^*)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2|a|^2 = 1 \Rightarrow a = \frac{1}{\sqrt{2}}$

[this is the inner product of $\begin{pmatrix} a \\ a \end{pmatrix}$ with itself]

$\therefore \hat{S}_x$: eigenvalue $+\frac{\hbar}{2}$ eigenvector

$$|\tfrac{1}{2}\;\tfrac{1}{2}\rangle_x = |\uparrow\rangle_x = \alpha_x \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

referring to $\hat{S}_x$ (23)

Do the same for eigenvalue $-\frac{\hbar}{2}$ (Ex.)

eigenvector is $|\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_x = |\downarrow\rangle_x = \beta_x \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$ (24)[†]

<u>How about $\hat{S}_y$?</u> [Do the same]

Eigenvalues are (again) $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

[as discussed under orbital AM, the components in <u>ANY</u> direction are the same]

$\hat{S}_y$: eigenvalue $+\frac{\hbar}{2}$ eigenvector $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_y = |\uparrow\rangle_y = \alpha_y \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$\hat{S}_y$: eigenvalue $-\frac{\hbar}{2}$ eigenvector $|\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_y = |\downarrow\rangle_y = \beta_y \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ (25)[†]

[†] α_x and β_x (for $\hat{S}_x$) are orthogonal to each other.
α_y and β_y (for $\hat{S}_y$) are orthogonal to each other.

<u>Summary</u>-

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}[\sigma_x]$$

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}[\sigma_y]$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}[\sigma_z]$$

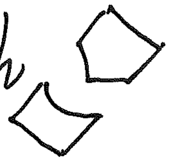
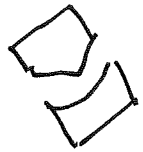
The eigenvalues and eigenvectors of the matrices representing the angular momentum components of a spin-half particle

Spin component	Eigenvalue	Eigenvector
$\hat{S}_x$	$\frac{1}{2}\hbar$	$\alpha_x \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\hat{S}_x$	$-\frac{1}{2}\hbar$	$\beta_x \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$\hat{S}_y$	$\frac{1}{2}\hbar$	$\alpha_y \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\hat{S}_y$	$-\frac{1}{2}\hbar$	$\beta_y \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$
$\hat{S}_z$	$\frac{1}{2}\hbar$	$\alpha_z \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\hat{S}_z$	$-\frac{1}{2}\hbar$	$\beta_z \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(26)

The point is: Any component of $\vec{S}$ $\underbrace{\text{in any}}_{\text{we did } x,\, y,\, z\text{-directions}}$ direction,

has eigenvalues $+\hbar/2$ and $-\hbar/2$

$\therefore$ Measure a component in any direction (place SG apparatus with  OR )

the outcome is either $+\dfrac{\hbar}{2}$ or $-\dfrac{\hbar}{2}$

[This is the quantum aspect of angular momentum]

(b) Using Spin to Practise Quantum Theory of Measurement

Recall: Measure A, outcomes are eigenvalues of $\hat{A}$

Given Ψ, $\quad \Psi = \sum_i c_i \phi_i \quad ; \quad \hat{A}\phi_i = a_i \phi_i$ $\quad$ (27)

$|c_i|^2 =$ Prob. of getting the outcome a_i

Behind the scene, measurements on identical copies, ...

Let's practice!

A general state of spin-½ particle can be expressed as a vector ("state vector") of the form

$$\begin{pmatrix} c \\ d \end{pmatrix} \qquad \underbrace{\text{(assumed normalized)}}_{\text{Meaning: } |c|^2 + |d|^2 = 1} \qquad (28)$$

Q1: Passing a beam of particles all prepared in state (28) into Stern-Gerlach (SG) experiment arranged to measure S_z (z-component) (Symbol: SGZ), what will happen?

Expand: $$\begin{pmatrix} c \\ d \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (29)$$

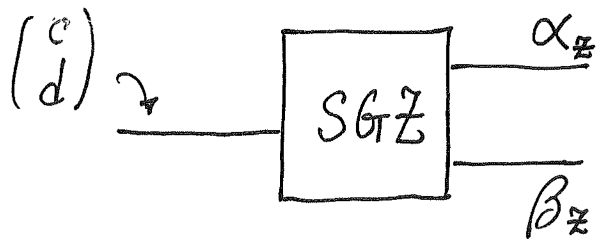
[$\Psi = c\,\alpha_z + d\,\beta_z$ c.f. $\Psi = \sum_i c_i \phi_i$ in (27)]

Slow Motion: $\underbrace{\text{Measure } S_z}$, look at $\hat{S}_z$ eigenvalues & eigenstates solved (see Eq. (26))

Expand $\begin{pmatrix} c \\ d \end{pmatrix}$ in terms of $\hat{S}_z$'s eigenstates (basis)

From (26), $|c|^2$ is the probability of getting S_z to be $+\frac{\hbar}{2}$

$|d|^2$ is the probability of getting S_z to be $-\frac{\hbar}{2}$



α_z collapsed to $\alpha_z = \binom{1}{0}$ probability $|c|^2$ getting $+\frac{\hbar}{2}$

β_z collapsed to $\beta_z = \binom{0}{1}$ probability $|d|^2$ getting $-\frac{\hbar}{2}$

[This also provides a way to prepare a beam with particles all in the same spin state, e.g. take one output beam.]

<u>Q2</u>: Same as in Q1, but apparatus arranged to measure <u>S_x (SGX)</u>

Expand $\binom{c}{d}$ in terms of α_x and β_x

eigenstates of $\hat{S}_x$ (see Eq. (26))

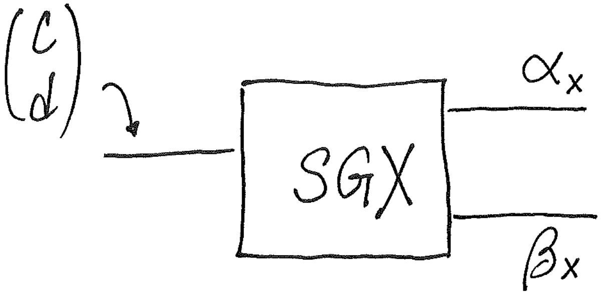
i.e. $$\begin{pmatrix} c \\ d \end{pmatrix} = c_1 \alpha_x + c_2 \beta_x = c_1 \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \; c_1 = \frac{c+d}{\sqrt{2}} \quad ; \quad c_2 = -\frac{(c-d)}{\sqrt{2}}$$

$$\therefore \begin{pmatrix} c \\ d \end{pmatrix} = \frac{c+d}{\sqrt{2}} \cdot \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{\alpha_x} - \frac{(c-d)}{\sqrt{2}} \cdot \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}^{\beta_x} \qquad (30)$$

From (30), $\frac{|c+d|^2}{2}$ is probability of getting S_x to be $+\frac{\hbar}{2}$

$\frac{|c-d|^2}{2}$ is probability of getting S_x to be $-\frac{\hbar}{2}$



collapsed to $\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ prob. $\frac{|c+d|^2}{2}$ getting $+\frac{\hbar}{2}$

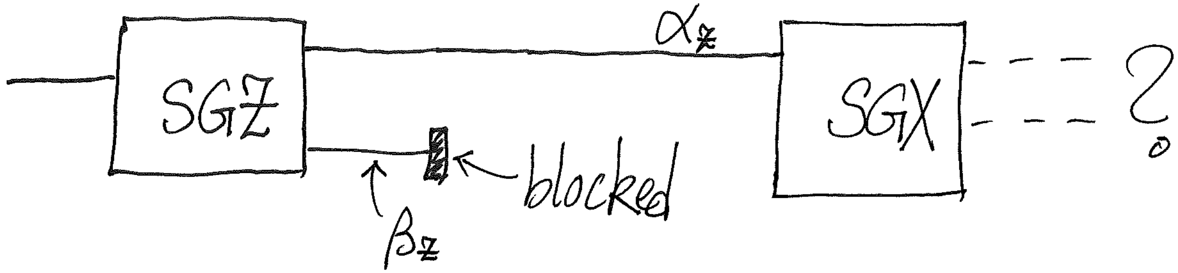
collapsed to $\beta_x = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ prob. $\frac{|c-d|^2}{2}$ getting $-\frac{\hbar}{2}$

<u>Note</u>: It is the same beam $\begin{pmatrix} c \\ d \end{pmatrix}$ to start with in Q1 & Q2, only what is being measured is different.
In both cases, $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are the possible outcomes.
But the probabilities depend on whether it is SGX OR SGZ.

Q3: Same as Q1 and Q2, but now arranged to measure S_y [i.e. SGY]. (Ex.)

(c) $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$ ⇒ State of definite S_z has no definite S_x, etc.

- Only when $[\hat{A}, \hat{B}] = 0$, then $\hat{A}, \hat{B}$ share common eigenstates, ... (Ch. X)



$\alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with definite $S_z = \frac{\hbar}{2}$ (after SGZ, select a beam)

Send $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ beam into SGX, what will happen?

[No new technique, just follow same technique]

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}_{\alpha_x} - \frac{1}{\sqrt{2}}\cdot\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}}_{\beta_x} \qquad (31)$$

definite $S_z = \frac{+\hbar}{2}$

$\therefore$ $\begin{cases} \text{Prob. } 1/2 \text{ of getting } S_x \text{ to be } +\frac{\hbar}{2} \\ \text{Prob. } 1/2 \text{ of getting } S_x \text{ to be } -\frac{\hbar}{2} \end{cases}$ $\langle S_z \rangle = 0$

Eq. (31) $\Rightarrow$ Eigenstate of $\hat{S}_z$ is NOT eigenstate of $\hat{S}_x$. $\left[\hat{S}_x\begin{pmatrix}1\\0\end{pmatrix} \neq \text{a number}\begin{pmatrix}1\\0\end{pmatrix}\right]$

(Recall $\hat{S}_z$ and $\hat{S}_x$ do not commute)

What if we take the $S_x = +\frac{\hbar}{2}$ output beam and do SGZ again?

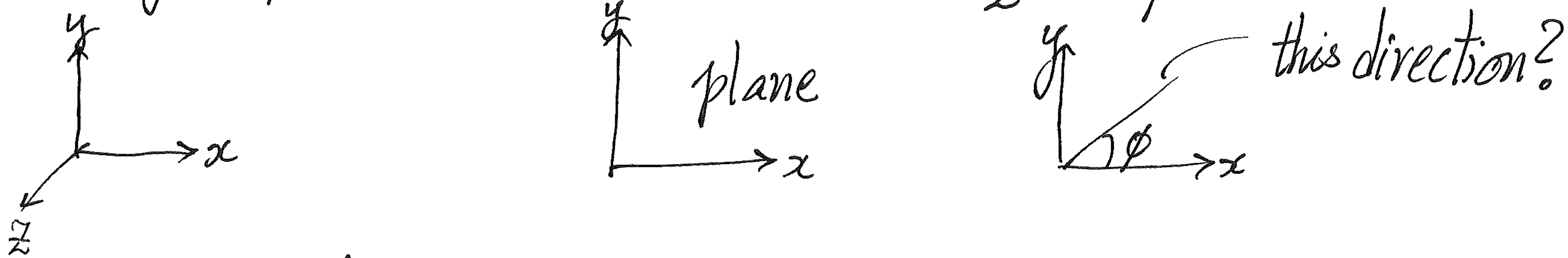
- $\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$

$\therefore$ No definite S_z prediction! (although started out with α_z)

Prob. 1/2 of getting $S_z = +\frac{\hbar}{2}$, Prob. 1/2 of getting $S_z = \frac{-\hbar}{2}$

- This can go on and on.
- These QM predictions have been tested over the years!
- Experiments can also be done by photons.

(d) Nothing Special about x, y, z-directions

- Along any direction, same $+\frac{\hbar}{2}$, $-\frac{\hbar}{2}$ components
- 

$$\hat{S}_\phi = \hat{S}_x \cos\phi + \hat{S}_y \sin\phi = \frac{\hbar}{2}\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \quad (32)$$

Ex: Show eigenvalues are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. Find eigenstates.

Remarks

- How about summing two momenta? (more advanced course)

  Orbital AM($\vec{L}$) + Spin AM($\vec{S}$) "Spin-Orbit interaction"

- Measurements using SG set up [e.g. SGϕ] is important in testing QM's predictions against some alternative proposals.

- Up/Down Spin
  left-hand/right-hand polarization (photon)
  0/1 states (information science)

  No wonder spin QM is at the heart of Quantum Information Science & Technology